Exposed Polynomials of $\mathcal{P}(2\mathbb{R}^2_{h(\frac{1}{2})})$

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Abstract: We show that every extreme polynomials of $\mathcal{P}(2\mathbb{R}^2_{h(\frac{1}{2})})$ is exposed.

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1. Introduction

According to the Krein-Milman Theorem, every nonempty convex set in a Banach space is fully described by the set of its extreme points. Let $n \in \mathbb{N}$. We write $B_E$ for the closed unit ball of a real Banach space $E$ and the dual space of $E$ is denoted by $E^*$. We recall that if $x \in B_E$ is said to be an extreme point of $B_E$ if $y, z \in B_E$ and $x = \lambda y + (1 - \lambda)z$ for some $0 < \lambda < 1$ implies that $x = y = z$. $x \in B_E$ is called an exposed point of $B_E$ if there is an $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of $B_E$ is an extreme point. We denote by $\text{ext} B_E$ and $\text{exp} B_E$ the sets of extreme and exposed points of $B_E$, respectively. We denote by $L^n(E)$ the Banach space of all continuous $n$-linear forms on $E$ endowed with the norm $\|T\| = \sup_{\|x\| = 1} |T(x_1, \ldots, x_n)|$. A $n$-linear form $T$ is symmetric if $T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every permutation $\sigma$ on $\{1, 2, \ldots, n\}$. We denote by $L^s_n(E)$ the Banach space of all continuous symmetric $n$-linear forms on $E$. A mapping $P : E \to \mathbb{R}$ is a continuous $n$-homogeneous polynomial if there exists a unique $T \in L^s_n(E)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. In this case it is convenient to write $T = \hat{P}$. We denote by $\mathcal{P}(n,E)$ the Banach space of all continuous $n$-homogeneous polynomials from $E$ into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\| = 1} |P(x)|$. Note that the spaces $L^n(E)$,

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$L_s^{(n)E}, P^{(n)E}$ are very different from a geometric point of view. In particular, for integral multilinear forms and integral polynomials one has ([2], [9], [42])

$$\text{ext} B_{L_s^{(n)E}} = \{ \phi_1 \phi_2 \cdots \phi_n : \phi_i \in \text{ext} B_{E^*} \},$$

$$\text{ext} B_{P_s^{(n)E}} = \{ \pm \phi^n : \phi \in E^*, ||\phi|| = 1 \},$$

where $L_s^{(n)E}$ and $P_s^{(n)E}$ are the spaces of integral $n$-linear forms and integral $n$-homogeneous polynomials on $E$, respectively. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [10].

Let us say about the stories of the classification problems of $\text{ext} B_X$ and $\text{exp} B_X$ if $X = P^{(n)E}$. Choi et al. ([4], [5]) initiated the classification problems and classified $\text{ext} B_X$ if $X = P^{(2t_p^2)}$ for $p = 1, 2$, where $t_p^2 = \mathbb{R}^2$ with the $l_p$-norm. B. Grecu [14] classified $\text{ext} B_X$ if $X = P^{(2t_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. Kim [18] classified $\text{exp} B_X$ if $X = P^{(2t_p^2)}$ for $1 \leq p \leq \infty$. Kim et al. [34] showed that every extreme $2$-homogeneous polynomials on a real separable Hilbert space is also exposed. Kim ([20], [26]) characterized $\text{ext} B_X$ and $\text{exp} B_X$ for $X = P^{(2d_s(1, w)^2)}$, where $d_s(1, w)^2 = \mathbb{R}^2$ with the octagonal norm

$$\|(x, y)\|_d := \max \left\{ |x|, |y|, \frac{|x| + |y|}{1 + w} : 0 < w < 1 \right\}.$$  

He showed [26] that $\text{ext} B_{P^{(2d_s(1, w)^2)}} \neq \text{exp} B_{P^{(2d_s(1, w)^2)}}$. In [31], Kim classified $\text{ext} B_X$ and using the classification of $\text{ext} B_X$, Kim computed the polarization and unconditional constants of the space $X$ if $X = P^{(2\mathbb{R}^2_{h(\frac{1}{2})})}$, where $\mathbb{R}^2_{h(w)}$ denotes the space $\mathbb{R}^2$ endowed with the hexagonal norm

$$\|(x, y)\|_{h(w)} := \max \{|y|, |x| + (1 - w)|y|\}.$$  

We refer to ([1]--[9], [11]--[43]) and references therein for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces.

We will denote by $T((x_1, y_1), (x_2, y_2)) = ax_1 x_2 + by_1 y_2 + c(x_1 y_2 + x_2 y_1)$ and $P(x, y) = ax^2 + by^2 + cxy$ a symmetric bilinear form and a 2-homogeneous polynomial on a real Banach space of dimension 2, respectively. Recently, Kim [31] classified the extreme points of the unit ball of $P^{(2\mathbb{R}^2_{h(\frac{1}{2})})}$ as follows:

$$\text{ext} B_{P^{(2\mathbb{R}^2_{h(\frac{1}{2})})}} = \left\{ \pm y^2, \pm (x^2 + \frac{1}{4} y^2 \pm xy), \pm (x^2 + \frac{3}{4} y^2), \right.$$

$$\pm [x^2 + (\frac{c^2}{4} - 1) y^2 \pm cxy],$$

$$\pm [c x^2 + (\frac{c + \sqrt{1 - c}}{4} - 1) y^2 \pm (c + 2\sqrt{1 - c}) xy] (0 \leq c \leq 1) \right\}.$$
In this paper, we show that that every extreme polynomials of \( P(2R^2_{h(1/2)}) \) is exposed.

2. Results

**Theorem 2.1.** ([31]) Let \( P(x, y) = ax^2 + by^2 + cxy \in P(2R^2_{h(1/2)}) \) with \( a \geq 0, c \geq 0 \) and \( a^2 + b^2 + c^2 \neq 0 \). Then:

Case 1: \( c < a \).

If \( a \leq 4b \), then
\[
\|P\| = \max \left\{ a, b, \left| 4a - b \right| + 4b \right\}.
\]

If \( a > 4b \), then
\[
\|P\| = \max \left\{ a, b, \left| 4a - b \right| + 4b \right\}.
\]

Case 2: \( c \geq a \).

If \( a \leq 4b \), then
\[
\|P\| = \max \left\{ a, b, \left| 4a - b \right| + 4b \right\}.
\]

If \( a > 4b \), then
\[
\|P\| = \max \left\{ a, b, \left| 4a - b \right| + 4b \right\}.
\]

**Theorem 2.2.** ([31])
\[
\text{ext}B_P(2R^2_{h(1/2)}) = \left\{ y^2, (x^2 + \frac{1}{4}y^2 \pm xy), (x^2 + \frac{3}{4}y^2), \right. \right.
\]
\[
\left. \left. \left. \pm \left[ x^2 + \left( \frac{c^2}{4} - 1 \right) y^2 \pm cxy \right], \right. \right. \right.
\]
\[
\left. \left. \left. \pm \left[ cx^2 + \left( c + 2\sqrt{1-c} \right) xy \right] \right. \right. \right. \right.
\]
\[
\left. \left. \left. \left( 0 \leq c \leq 1 \right) \right. \right. \right. \right).
\]

**Theorem 2.3.** Let \( f \in P(2R^2_{h(1/2)})^* \) with \( \alpha = f(x^2), \beta = f(y^2), \gamma = f(xy) \). Then
\[
\|f\| = \sup \left\{ |\beta|, |\alpha + \frac{1}{3}\beta| + |\gamma|, |\alpha + \frac{3}{4}\beta|, |\alpha + \frac{c^2}{4} - 1\beta| + c|\gamma|, \right.
\]
\[
\left. |\alpha + \frac{c^2}{4} - 1\beta| + (c + 2\sqrt{1-c})|\gamma| \right\} \left( 0 \leq c \leq 1 \right).
\]

**Proof.** It follows from Theorem 2.2 and the fact that
\[
\|f\| = \sup \left\{ |f(P)| : P \in \text{ext}B_P(2R^2_{h(1/2)}) \right\}.
\]
Note that if \( \|f\| = 1 \), then \( |\alpha| \leq 1, |\beta| \leq 1, |\gamma| \leq \frac{1}{2} \).

We are in a position to show the main result of this paper.

**Theorem 2.4.**

\[
\exp B_p(2R^2_{1/2}) = \text{ext} B_p(2R^2_{1/2}).
\]

**Proof.** Let \( 0 \leq c \leq 1 \)

\[
P_1(x, y) = y^2,
\]

\[
P_2^+(x, y) = x^2 + \frac{1}{4}y^2 + xy,
\]

\[
P_2^-(x, y) = x^2 + \frac{1}{4}y^2 - xy,
\]

\[
P_3(x, y) = x^2 + \frac{3}{2}y^2,
\]

\[
P_4^+(c, x, y) = x^2 + (\frac{c^2}{4} - 1)y^2 + cxy,
\]

\[
P_4^-(c, x, y) = x^2 + (\frac{c^2}{4} - 1)y^2 - cxy,
\]

\[
P_5^+(c, x, y) = cx^2 + (\frac{c^2 + 4\sqrt{1-c}}{4} - 1)y^2 + (c + 2\sqrt{1-c})xy,
\]

\[
P_5^-(c, x, y) = cx^2 + (\frac{c^2 + 4\sqrt{1-c}}{4} - 1)y^2 - (c + 2\sqrt{1-c})xy.
\]

Claim 1: \( P_1 = y^2 \in \exp B_p(2R^2_{1/2}) \);

Let \( f \in \mathcal{P}(2R^2_{1/2})^* \) be such that

\[
\alpha = \frac{1}{5}, \quad \beta = 1, \quad \gamma = 0.
\]

Indeed,

\[
f(P_1) = 1, \quad |f(P_2^+)| = \frac{9}{20}, \quad |f(P_3)| = \frac{19}{20}.
\]

\[\]

Note that for all \( 0 \leq c \leq 1 \),

\[
|f(P_4^\pm)| = \frac{4}{5} - \frac{c^2}{4} \leq \frac{4}{5}, \quad (**)
\]

\[
|f(P_5^\pm)| = |\sqrt{1-c} + \frac{9c}{20} - 1| \leq \frac{11}{20}.
\]

\[\]
Hence, by Theorem 2.3, 1 = \|f\|. We will show that f exposes P. Let 
Q(x, y) = ax^2 + by^2 + cxy \in \mathcal{P}(\mathbb{R}^2_{\mathbb{R}^2_{h(\frac{1}{2})}}) \) such that 1 = \|Q\| = f(Q). We will 
show that Q = P. Since \mathcal{P}(\mathbb{R}^2_{\mathbb{R}^2_{h(\frac{1}{2})}}) is a finite dimensional Banach space with 
dimension 3, by the Krein-Milman Theorem, \text{B}_{\mathcal{P}(\mathbb{R}^2_{\mathbb{R}^2_{h(\frac{1}{2})}})} is the closed convex 
hull of \text{ext}\mathcal{B} \mathcal{P}(\mathbb{R}^2_{\mathbb{R}^2_{h(\frac{1}{2})}}). Then,
\begin{align*}
Q(x, y) & = uP_1(x, y) + v^+P_2^+(x, y) + v^-P_2^-(x, y) + tP_3(x, y) \\
& + \sum_{n=1}^{\infty} \lambda_n^+ P_{4,cn}^+(x, y) + \sum_{n=1}^{\infty} \lambda_n^- P_{4,cn}^-(x, y) \\
& + \sum_{m=1}^{\infty} \delta_m^+ P_{5,am}^+(x, y) + \sum_{m=1}^{\infty} \delta_m^- P_{5,am}^-(x, y),
\end{align*}
for some u, v^+, t, \lambda_n^\pm, \delta_m^\pm, \in \mathbb{R} (n, m \in \mathbb{N}) with 0 \leq c_n, a_m \leq 1 and
|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda_n^+| + \sum_{n=1}^{\infty} |\lambda_n^-| + \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| = 1.

We will show that v^+ = t = \lambda_n^\pm = \delta_m^\pm = 0 for every n, m \in \mathbb{N}.

Subclaim: v^+ = t = 0.
Assume that v^+ \neq 0. It follows that
\begin{align*}
1 & = f(Q) = uf(P_1) + v^+f(P_2^+) + v^-f(P_2^-) + tf(P_3) + \sum_{n=1}^{\infty} \lambda_n^+ f(P_{4,cn}^+) \\
& + \sum_{n=1}^{\infty} \lambda_n^- f(P_{4,cn}^-) + \sum_{m=1}^{\infty} \delta_m^+ f(P_{5,am}^+) + \sum_{m=1}^{\infty} \delta_m^- f(P_{5,am}^-) \\
& \leq |u| + |v^+||f(P_2^+)| + |v^-||f(P_2^-)| + |t||f(P_3)| + \sum_{n=1}^{\infty} |\lambda_n^+||f(P_{4,cn}^+)| \\
& + \sum_{n=1}^{\infty} |\lambda_n^-||f(P_{4,cn}^-)| + \sum_{m=1}^{\infty} |\delta_m^+||f(P_{5,am}^+)| + \sum_{m=1}^{\infty} |\delta_m^-||f(P_{5,am}^-)| \\
& \leq |u| + \frac{9}{20} |v^+| + \frac{9}{20} |v^-| + \frac{19}{20} |t| + \frac{4}{5} \sum_{n=1}^{\infty} |\lambda_n^+| \\
& + \frac{4}{5} \sum_{n=1}^{\infty} |\lambda_n^-| + \frac{11}{20} \sum_{m=1}^{\infty} |\delta_m^+| + \frac{11}{20} \sum_{m=1}^{\infty} |\delta_m^-| \quad \text{(by (**), (**)), (***)}
\end{align*}
< |u| + |v^+| + 9/20 |v^-| + 19/20 |t| + \frac{4}{5} \sum_{n=1}^{\infty} |\lambda_n^+| \\
\quad + \frac{4}{5} \sum_{n=1}^{\infty} |\lambda_n^-| + \frac{11}{20} \sum_{m=1}^{\infty} |\delta_m^+| + \frac{11}{20} \sum_{m=1}^{\infty} |\delta_m^-| \\
\leq |u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda_n^+| + \sum_{n=1}^{\infty} |\lambda_n^-| + \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| = 1,

which is impossible. Therefore, v^+ = 0. Using a similar argument as above, we have v^- = t = 0.

Subclaim: \lambda_n^+ = \delta_m^+ = 0 for every n, m \in \mathbb{N}.

Assume that \lambda_{n_0}^+ \neq 0 for some n_0 \in \mathbb{N}. It follows that

\begin{align*}
1 &= f(Q) = uf(P_1) + \lambda_{n_0}^- f(P_{4,c_{n_0}}) + \sum_{n \in \mathbb{N}, n \neq n_0} \lambda_n^- f(P_{4,c_n}^+) \\
&\quad + \sum_{n=1}^{\infty} \lambda_n^- f(P_{4,c_n}^-) + \sum_{m=1}^{\infty} \delta_m^+ f(P_{5,a_m}^+) + \sum_{m=1}^{\infty} \delta_m^- f(P_{5,a_m}^-) \\
\leq |u| + |\lambda_{n_0}^-||f(P_{4,c_{n_0}}^+)| + \sum_{n \in \mathbb{N}, n \neq n_0} |\lambda_n^-||f(P_{4,c_n}^+)| + \sum_{n=1}^{\infty} |\lambda_n^-||f(P_{4,c_n}^-)| \\
&\quad + \sum_{m=1}^{\infty} |\delta_m^+||f(P_{5,a_m}^+)| + \sum_{m=1}^{\infty} |\delta_m^-||f(P_{5,a_m}^-)| \\
\leq |u| + |\lambda_{n_0}^-| + \frac{4}{5} \sum_{n \in \mathbb{N}, n \neq n_0} |\lambda_n^+| + \frac{11}{20} \sum_{n=1}^{\infty} |\lambda_n^-| + \frac{11}{20} \sum_{m=1}^{\infty} |\delta_m^+| + \frac{11}{20} \sum_{m=1}^{\infty} |\delta_m^-| \\
\leq |u| + \sum_{n=1}^{\infty} |\lambda_n^+| + \sum_{n=1}^{\infty} |\lambda_n^-| + \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| = 1,
\end{align*}

which is impossible. Therefore, \lambda_n^+ = 0 for every n \in \mathbb{N}. Using a similar argument as above, we have \lambda_n^- = \delta_m^- = 0 for every n, m \in \mathbb{N}. Therefore, Q(x, y) = uP_1(x, y). Hence u = 1, so Q = P_1. Therefore, f exposes P_1.

Claim 2: P_{5,0} = 2xy \in \exp B_{P}(\mathbb{R}^2_{h(\frac{1}{2})}).

Let f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})^* be such that

\alpha = \beta = 0, \quad \gamma = \frac{1}{2}.
We will show that $f$ exposes $P_{5,0}$. Indeed, $f(P_{5,0}) = 1$, $f(P_1) = 0$, $f(P_2) = \pm \frac{1}{2}$, $f(P_3) = 0$,

$$-\frac{1}{2} \leq f(P_{4,c}) = \pm \frac{c}{2} \leq \frac{1}{2} \quad (0 \leq c \leq 1).$$

Note that, for $0 < c \leq 1$,

$$-1 < f(P_{5,c}) = \pm \frac{c + 2\sqrt{1-c}}{2} < 1. \quad (\dagger)$$

Hence, by Theorem 2.3, $1 = \|f\|$. Let

$$Q(x, y) = uP_1(x, y) + v^+ P_2^+(x, y) + v^- P_2^-(x, y) + tP_3(x, y)$$

$$+ \sum_{n=1}^{\infty} \lambda^+_n P_{4,c}^+(x, y) + \sum_{n=1}^{\infty} \lambda^-_n P_{4,c}^-(x, y)$$

$$+ \sum_{m=1}^{\infty} \delta^+_m P_{5,a_m}^+(x, y) + \sum_{m=1}^{\infty} \delta^-_m P_{5,a_m}^-(x, y),$$

for some $u, v^+, t, \lambda^+_n, \delta^+_m \in \mathbb{R}$ ($n, m \in \mathbb{N}$) with $0 \leq c^+_n, a^+_m \leq 1$ and

$$|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda^+_n| + \sum_{n=1}^{\infty} |\lambda^-_n| + \sum_{m=1}^{\infty} |\delta^+_m| + \sum_{m=1}^{\infty} |\delta^-_m| = 1.$$ 

We will show that $v^\pm = t = \lambda^+_n = \delta^+_m = 0$ for every $n, m \in \mathbb{N}$.

Subclaim: $v^+ = 0$.

Assume that $v^+ \neq 0$. It follows that

$$1 = f(Q) = v^+ f(P_2^+) + v^- f(P_2^-) + \sum_{n=1}^{\infty} \lambda^+_n f(P_{4,c}^+)$$

$$+ \sum_{n=1}^{\infty} \lambda^-_n f(P_{4,c}^-) + \sum_{m=1}^{\infty} \delta^+_m f(P_{5,a_m}^+) + \sum_{m=1}^{\infty} \delta^-_m f(P_{5,a_m}^-)$$

$$< |v^+| + \frac{1}{2} |v^-| + \sum_{n=1}^{\infty} |\lambda^+_n||f(P_{4,c}^+)| + \sum_{n=1}^{\infty} |\lambda^-_n||f(P_{4,c}^-)|$$

$$+ \sum_{m=1}^{\infty} |\delta^+_m||f(P_{5,a_m}^+)| + \sum_{m=1}^{\infty} |\delta^-_m||f(P_{5,a_m}^-)|$$

$$\leq |v^+| + |v^-| + \sum_{n=1}^{\infty} |\lambda^+_n| + \sum_{n=1}^{\infty} |\lambda^-_n| + \sum_{m=1}^{\infty} |\delta^+_m| + \sum_{m=1}^{\infty} |\delta^-_m| \leq 1,$$
which is impossible. Therefore, $v^+ = 0$. Using a similar argument as Claim 1, we have $v^- = \lambda_n^+ = 0$ for every $n \in \mathbb{N}$. Hence,

$$Q(x, y) = uP_1(x, y) + tP_3(x, y) + \sum_{m=1}^{\infty} \delta_m^+ P_{5,a_m}^+(x, y) + \sum_{m=1}^{\infty} \delta_m^- P_{5,a_m}^-(x, y).$$

It follows that

$$1 = f(Q) = \sum_{m=1}^{\infty} \delta_m^+ f(P_{5,a_m}^+) + \sum_{m=1}^{\infty} \delta_m^- f(P_{5,a_m}^-)$$

$$\leq \sum_{m=1}^{\infty} |\delta_m^+| |f(P_{5,a_m}^+)| + \sum_{m=1}^{\infty} |\delta_m^-| |f(P_{5,a_m}^-)|$$

$$\leq \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| \leq 1,$$

which shows that

$$f(P_{5,a_m}^+) = f(P_{5,a_m}^-) = \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| = 1, \quad u = t = 0 \quad \text{for all } m \in \mathbb{N}.$$

By (\dagger), $P_{5,a_m}^\pm = P_{5,0}^\pm$ for every $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} \delta_m^+ + \sum_{m=1}^{\infty} \delta_m^- = 1$. Therefore, $Q = P_{5,0}$. Hence, $f$ exposes $P_{5,0}$.

Claim 3: $P_2^+ = x^2 + \frac{1}{4} y^2 + xy \in \exp B_p(2\mathbb{R}^2_{h(\frac{1}{2})})$.

Let $f \in \mathcal{P}(2\mathbb{R}^2_{h(\frac{1}{2})})^*$ be such that

$$\alpha = \frac{1}{2} = \beta, \quad \gamma = \frac{3}{8}.$$

We will show that $f$ exposes $P_2$. Indeed, $f(P_2^+) = 1$, $f(P_2^-) = \frac{1}{4}$, $f(P_1) = \frac{1}{2}$, $f(P_3^\pm) = \frac{7}{8}$. By some calculation, we have

$$|f(P_{4,c}^\pm)| \leq \frac{1}{2}, \quad |f(P_{5,c}^\pm)| \leq \frac{57}{64} \quad \text{for } 0 \leq c \leq 1.$$

Hence, by Theorem 2.3, $1 = \|f\|$. By similar arguments as Claims 1 and 2, $f$ exposes $P_2^+$. Obviously, $P_2^- \in \exp B_p(2\mathbb{R}^2_{h(\frac{1}{2})})$.

Claim 4: $P_{4,0}^+ = x^2 - y^2 \in \exp B_p(2\mathbb{R}^2_{h(\frac{1}{2})})$. 
Let \( f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})^* \) be such that
\[
\alpha = \frac{1}{2} = -\beta, \quad \gamma = 0.
\]
We will show that \( f \) exposes \( P_{4,0} \). Indeed,
\[
f(P_{4,0}^+) = 1, \quad |f(P_1^+)| = \frac{1}{2}, \quad |f(P_2^+) - 3| = \frac{3}{8}, \quad |f(P_0)| = \frac{1}{8}.
\]
Note that
\[
|f(P_{4,c}^+)| = 1 - \frac{c^2}{8} < 1 \quad \text{for } 0 < c \leq 1.
\]
Note that, for \( 0 \leq c \leq 1 \),
\[
|f(P_{5,c}^+)| = \frac{3c + 4 - 4\sqrt{1-c}}{8} \leq \frac{7}{8}.
\]
Hence, by Theorem 2.3, \( 1 = \|f\| \). By similar arguments as Claims 1 and 2, \( f \) exposes \( P_{4,0}^+ \).

Claim 5: \( P_3 = x^2 + 4\frac{3}{4}y^2 \in \exp B_P(x^{2h}_{h(\frac{1}{2})}) \).

Let \( f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})^* \) be such that
\[
\alpha = \frac{5}{8}, \quad \beta = \frac{1}{2}, \quad \gamma = 0.
\]
We will show that \( f \) exposes \( P_3 \). Indeed,
\[
f(P_3) = 1, \quad |f(P_1)| = \frac{1}{2}, \quad |f(P_2^+) - 3| = \frac{3}{4}.
\]
Note that
\[
|f(P_{4,c}^+)| \leq \frac{1}{4}, \quad |f(P_{5,c}^+)| \leq \frac{1}{3} \quad \text{for } 0 \leq c \leq 1.
\]
Hence, by Theorem 2.3, \( 1 = \|f\| \). By similar arguments as Claims 1 and 2, \( f \) exposes \( P_3 \).

Claim 6: \( P_{5,1}^+ = x^2 - 3y^2 + xy \in \exp B_P(x^{2h}_{h(\frac{1}{2})}) \).

Let \( f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})^* \) be such that
\[
\alpha = \frac{11}{16}, \quad \beta = -\frac{1}{4}, \quad \gamma = \frac{1}{8}.
\]
We will show that \( f \) exposes \( P_{5,1}^+ \). Indeed,

\[
f(P_{5,1}^+) = 1, \quad |f(P_1)| = \frac{1}{4}, \quad |f(P_2^+)| \leq \frac{3}{4}, \quad |f(P_3)| = \frac{1}{2}.
\]

Note that

\[
\frac{3}{4} \leq f(P_{5,c}^+) < 1, \quad -\frac{1}{4} \leq f(P_{5,c}^-) < 1 \quad \text{for } 0 \leq c < 1.
\]

Hence, by Theorem 2.3, \( 1 = \|f\| \). By similar arguments as Claims 1 and 2, \( f \) exposes \( P_{5,1}^+ \). Obviously, \( P_{5,1}^- \in \exp_B \( P_{2,1}^+ \) \).

Claim 7: \( P_{4,c}^+ = x^2 + (\frac{c^2}{4} - 1)y^2 + cxy \in \exp_B \( P_{2,1}^+ \) \) for \( 0 < c < 1 \).

Let \( f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})* \) be such that

\[
\alpha = \frac{3}{4} - \frac{c^2}{16}, \quad \beta = -\frac{1}{4}, \quad \gamma = \frac{c}{8}.
\]

Indeed,

\[
f(P_{4,c}^+) = 1, \quad \frac{3}{4} \leq f(P_{4,c}^-) = 1 - \frac{c^2}{4} < 1, \quad |f(P_1)| = \frac{1}{4}, \quad (\ast)
\]

\[
\frac{1}{2} \leq f(P_2^+) \leq \frac{3}{4}, \quad \frac{1}{2} \leq f(P_3) < \frac{9}{16}.
\]

Note that for every \( t \in [0, 1] \) with \( t \neq c \),

\[
f(P_{4,t}^+) = -\frac{1}{16}t^2 + \frac{c}{8}t + \left(1 - \frac{c^2}{16}\right)
\]

and

\[
f(P_{4,t}^-) = -\frac{1}{16}t^2 - \frac{c}{8}t + \left(1 - \frac{c^2}{16}\right).
\]

Hence, we have, for every \( t \in [0, 1] \) with \( t \neq c \),

\[
1 < \min \left\{ 1 - \frac{c^2}{16}, 1 - \frac{(1-c)^2}{16} \right\} \leq f(P_{4,t}^+) < 1 \quad (\ast\ast)
\]

and

\[
-1 < 1 - \frac{(1+c)^2}{16} \leq f(P_{4,t}^-) \leq 1 - \frac{c^2}{16} < 1.
\]
exposed polynomials of $P(2\mathbb{R}^2_{h(\frac{1}{2})})$

Note that, for every $t \in [0, 1],$

$$f(P_{5,t}^+) = \left(-\frac{c^2 + 2c + 11}{16}\right) t + \left(\frac{c}{4} - 1\right) \sqrt{1 - t} + \frac{1}{4}$$

and

$$f(P_{5,t}^-) = \left(-\frac{c^2 - 2c + 11}{16}\right) t + \left(\frac{c + 1}{4}\right) \sqrt{1 - t} + \frac{1}{4}.$$  

Hence, we have that, for every $t \in [0, 1],$

$$-1 < \frac{c}{4} \leq f(P_{5,t}^+) \leq \frac{-c^2 + 2c + 15}{16} < 1 \quad (***$$

and

$$-1 < \frac{c + 2}{4} \leq f(P_{5,t}^-) \leq \frac{-c^2 - 2c + 15}{16} < 1.$$  

Hence, by Theorem 2.3, $1 = \|f\|$. We will show that $f$ exposes $P_{4,c}^+$. Let $Q(x, y) = ax^2 + by^2 + cxy \in P(2\mathbb{R}^2_{h(\frac{1}{2})})$ such that $1 = \|Q\| = f(Q)$. We will show that $Q = P_{4,c}^+$. By the Krein-Milman Theorem,

$$Q(x, y) = uP_1(x, y) + v^+P_2^+(x, y) + v^-P_2^-(x, y) + tP_3(x, y)$$

$$+ \sum_{n=1}^{\infty} \lambda_n^+ P_{4,c_n^+}^+(x, y) + \sum_{n=1}^{\infty} \lambda_n^- P_{4,c_n^-}^-(x, y)$$

$$+ \sum_{m=1}^{\infty} \delta_m^+ P_{5,a_m^+}^+(x, y) + \sum_{m=1}^{\infty} \delta_m^- P_{5,a_m^-}^-(x, y),$$

for some $u, v^+, v^-, t, \lambda_n^+, \delta_m^+ \in \mathbb{R}$ $(n, m \in \mathbb{N})$ with $0 \leq c_n^+, a_m^+ \leq 1$ and

$$|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda_n^+| + \sum_{n=1}^{\infty} |\lambda_n^-| + \sum_{m=1}^{\infty} |\delta_m^+| + \sum_{m=1}^{\infty} |\delta_m^-| = 1.$$  

We will show that $u = v^+ = t = \lambda_n^- = \delta_m^- = 0$ for every $n, m \in \mathbb{N}$. Assume
that $\delta_{m_0}^+ \neq 0$ for some $m_0 \in \mathbb{N}$. It follows that

$$1 = f(Q) = uf(P_1) + v^+ f(P_2^+) + v^- f(P_2^-) + tf(P_3) + \sum_{n=1}^{\infty} \lambda^+_n f(P_{4,c_n}^+),$$

$$+ \sum_{n=1}^{\infty} \lambda^-_n f(P_{4,c_n}^-) + \sum_{m=1}^{\infty} \delta^+_m f(P_{5,a_m}^+) + \sum_{m=1}^{\infty} \delta^-_m f(P_{5,a_m}^-)$$

$$< \frac{1}{4} |u| + \frac{3}{4} |v^+| + \frac{3}{4} |v^-| + \frac{9}{10} |t| + \sum_{n=1}^{\infty} |\lambda^+_n|$$

$$+ \sum_{n=1}^{\infty} |\lambda^-_n| + |\delta^+_m| + |\delta^-_m| \quad \text{(by (**), (**)), (***)} \leq 1,$$

which is impossible. Therefore, $\delta_m^+ = 0$ for every $m \in \mathbb{N}$. Using a similar argument as above, we have $u = v^\pm = t = \lambda^- = 0$. Therefore,

$$Q(x, y) = \sum_{n=1}^{\infty} \lambda_n^+ P_{4,c_n}^+(x, y).$$

We will show that if $c_{n_0}^+ \neq c$ for some $n_0 \in \mathbb{N}$, then $\lambda_{n_0}^+ = 0$. Assume that $\lambda_{n_0}^+ \neq 0$. It follows that

$$1 = f(Q) = \lambda_{n_0}^+ f(P_{4,c_{n_0}^+}^+) + \sum_{n \neq n_0} \lambda_n^+ f(P_{4,c_n}^+)$$

$$< |\lambda_{n_0}^+| + \sum_{n \neq n_0} |\lambda_n^+| = 1,$$

which is impossible. Therefore, $\lambda_n^+ = 0$ for every $n \in \mathbb{N}$. Therefore,

$$Q(x, y) = \left( \sum_{c_n^+ = c} \lambda_n^+ \right) P_{4,c}^+(x, y) = P_{4,c}^+(x, y).$$

Therefore, $f$ exposes $P_{4,c}^+$. Obviously, $P_{4,c}^+ \in \text{expB}_{\mathcal{P} \left( z_{\mathbb{R}^2}^{2h(1/2)} \right)}$ for $0 < c \leq 1$.

Claim 8: $P_{5,c}^+ = cx^2 + \left( \frac{c+4\sqrt{1-c} \gamma}{4} - 1 \right) y^2 + (c+2\sqrt{1-c})xy \in \text{expB}_{\mathcal{P} \left( z_{\mathbb{R}^2}^{2h(1/2)} \right)}$ for $0 < c < 1$. 
Let $f \in \mathcal{P}(\mathbb{R}^2_{h(\frac{1}{2})})^*$ be such that

$$
\alpha = \frac{1}{2} \left(1 - \frac{c + 4\sqrt{1-c}}{4}\right), \quad \beta = -\frac{c}{2}, \quad \gamma = \frac{c + 2\sqrt{1-c}}{4}.
$$

Note that

$$0 \leq \alpha < \frac{3}{8}, \quad -\frac{1}{2} < \beta \leq 0, \quad \frac{1}{4} < \gamma \leq \frac{1}{2}.$$

We will show that $f$ exposes $P_{5c}^+$. Indeed,

$$f(P_{5c}^+) = 1, \quad |f(P_1)| < \frac{1}{2}, \quad 0 < f(P_2^+) < \frac{1}{2},$$

$$-1 < f(P_2^-) < -\frac{1}{8}, \quad -\frac{1}{8} \leq f(P_3) < 0. \quad (*)$$

Note that for every $t \in [0, 1]$,

$$f(P_{4t}^+) = -\frac{c}{8}t^2 + \left(\frac{c + 2\sqrt{1-c}}{4}\right)t + \frac{1}{2} + \frac{3c}{8} - \frac{\sqrt{1-c}}{2}$$

and

$$f(P_{4t}^-) = -\frac{c}{8}t^2 - \left(\frac{c + 2\sqrt{1-c}}{4}\right)t + \frac{1}{2} + \frac{3c}{8} - \frac{\sqrt{1-c}}{2}.$$

Hence, we have for every $t \in [0, 1]$,

$$-1 < \frac{1}{2} + \frac{3c}{8} - \frac{\sqrt{1-c}}{2} \leq f(P_{4t}^+) \leq \frac{c + 1}{2} < 1, \quad (***)$$

$$-1 < \frac{1}{2} - \sqrt{1-c} \leq f(P_{4t}^-) \leq \frac{1}{2} + \frac{3c}{8} - \frac{\sqrt{1-c}}{2} < 1.$$

Note that for every $t \in [0, 1]$ with $t \neq c$,

$$f(P_{5t}^+) = \frac{1}{2}t + \sqrt{1-c} \sqrt{1-t} + \frac{c}{2}$$

and

$$f(P_{5t}^-) = \left(\frac{1 - c - \sqrt{1-c}}{2}\right)t - (c + \sqrt{1-c}) \sqrt{1-t} + \frac{c}{2}.$$

Hence, we have for every $t \in [0, 1]$ with $t \neq c$,

$$-1 < \min \left\{\frac{c}{2} + \sqrt{1-c}, \frac{c + 1}{2}\right\} \leq f(P_{5t}^+) < 1, \quad (***)$$

$$-1 < -\left(\frac{c}{2} + \sqrt{1-c}\right) \leq f(P_{5t}^-) \leq \frac{1}{2} - \sqrt{1-c} < 1.$$
Hence, by Theorem 2.3, \( 1 = \|f\| \). Let \( Q(x, y) = ax^2 + by^2 + cxy \) in \( \mathcal{P}(\mathbb{R}^2) \) such that \( 1 = \|Q\| = f(Q) \). By the Krein-Milman Theorem,

\[
Q(x, y) = u P_1(x, y) + v^+ P^+_2(x, y) + v^- P^-_2(x, y) + t P_3(x, y)
\]

\[
\quad + \sum_{n=1}^{\infty} \lambda^+_n P^+_{4,c_n}(x, y) + \sum_{n=1}^{\infty} \lambda^-_n P^-_{4,c_n}(x, y)
\]

\[
\quad + \sum_{m=1}^{\infty} \delta^+_m P^+_{5,a_m}(x, y) + \sum_{m=1}^{\infty} \delta^-_m P^-_{5,a_m}(x, y),
\]

for some \( u, v, \lambda^+_n, \lambda^-_n, \delta^+_m, \delta^-_m \in \mathbb{R} \) with \( 0 \leq c_n, a_m \leq 1 \) and

\[
|u| + |v^+| + |v^-| + |t| + \sum_{n=1}^{\infty} |\lambda^+_n| + \sum_{n=1}^{\infty} |\lambda^-_n| + \sum_{m=1}^{\infty} |\delta^+_m| + \sum_{m=1}^{\infty} |\delta^-_m| = 1.
\]

We will show that \( u = v^+ = t = \lambda^+_n = \delta^-_m = 0 \) for every \( n, m \in \mathbb{N} \). Assume that \( \lambda_n \neq 0 \) for some \( n_0 \in \mathbb{N} \). It follows that

\[
1 = f(Q) = uf(P_1) + v^+ f(P^+_2) + v^- f(P^-_2) + tf(P_3) + \sum_{n=1}^{\infty} \lambda^+_n f(P^+_{4,c_n})
\]

\[
\quad + \sum_{n=1}^{\infty} \lambda^-_n f(P^-_{4,c_n}) + \sum_{m=1}^{\infty} \delta^+_m f(P^+_{5,a_m}) + \sum_{m=1}^{\infty} \delta^-_m f(P^-_{5,a_m})
\]

\[
< \frac{1}{2} |u| + \frac{1}{2} |v^+| + \frac{1}{2} |v^-| + \frac{1}{2} |t| + |\lambda^+_n| + \sum_{n \neq n_0} |\lambda^-_n| + \sum_{m=1}^{\infty} |\delta^+_m| + \sum_{m=1}^{\infty} |\delta^-_m|
\]

\[
\leq 1 \quad \text{(by (**), (***)�)},
\]

which is impossible. Therefore, \( \lambda^+_n = 0 \) for every \( n \in \mathbb{N} \). Using a similar argument as above, we have \( u = v^+ = t = \lambda^-_n = \delta^-_m = 0 \) for every \( n, m \in \mathbb{N} \). Therefore,

\[
Q(x, y) = \sum_{m=1}^{\infty} \delta^+_m P^+_{5,a_m}(x, y).
\]

We will show that if \( a^+_{m_0} \neq c \) for some \( m_0 \in \mathbb{N} \), then \( \delta^+_{m_0} = 0 \). Assume that
\[ \delta_{m_0}^+ \neq 0. \] It follows that
\[ 1 = f(Q) = \delta_{m_0}^+ f(P_{5,a_{m_0}}^+) + \sum_{m \neq m_0} \delta_m^+ f(P_{5,a_m}^+) \]
\[ < |\delta_{m_0}^+| + \sum_{m \neq m_0} |\delta_m^+| = 1 \]
which is impossible. Therefore, \( \delta_{m_0}^+ = 0. \) Therefore,
\[ Q(x, y) = \left( \sum_{a_m = 0} \delta_m^+ \right) P_{5,c}^+(x, y) = P_{5,c}^+(x, y). \]
Therefore, \( f \) exposes \( P_{5,c}^+ \). Obviously, \( P_{5,c}^+ \in \exp B_{p}(\mathbb{R}_h^2_{\cdot 1/2}) \) for \( 0 < c < 1. \) Therefore, we complete the proof. \( \blacksquare \)

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